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Topological charges and the genus of surfaces

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Abstract

We show that the topological charge of the n-soliton solution of the sine-Gordon equation $n[\phi] = (1/2\pi)[\int \mathrm{d}x\partial_x\phi]$ is related to the genus $g > 1$ of a constant negative curvature compact orientable surface described by this configuration. The relation is $n = 2(g-1)$, where $n = 2\nu$ is even. The moduli space of complex dimension $B_g = 3(g-1)$ corresponds precisely to the freedom to choose the configuration with n solitons of arbitrary positions and velocities. We speculate also that the odd soliton states will described the unoriented surfaces.

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1. Introduction

The sine-Gordon equation [1,2]

$$\Phi_{tt} - \Phi_{xx} = -\sin\Phi, \tag{1.1}$$

enjoys a great importance in physics:

(a) In the lagrangian formalism it presents *spontaneous breaking* of the discrete Z symmetry $\Phi \to \Phi + 2\pi k$ and exhibits the attendant soliton and multisoliton phenomenon. The

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single static soliton is given as solution of the first-order (Bogomolny) equation; if $V(\Phi) = \frac{1}{2}W^2(\Phi)$,

$$\Phi'(x) \equiv W(\Phi) = 2(\sin \tfrac{1}{2}\Phi) \tag{1.2}$$

has as solution the profile

$$\Phi(x) = 4 \operatorname{arctg} \exp(x - x_0). \tag{1.3}$$

(b) The sine-Gordon equation has *auto-Bäcklund transformation* [3] which makes it possible to obtain the general n-soliton solution [4]. The theory is exactly soluble also by the inverse spectral transform method [5]. The general solution contains multiple soliton/antisoliton configurations, as well as breathers (soliton–antisoliton bound states) and background "noise". The quantization is factible [6,7] and indeed the formulas of the WKB approximation are already exact [8].

(c) The *quantized* sine-Gordon theory is equivalent to the massive Thirring model [9]; in fact, this duality, already conjectured by Skyrme [10], is the first case found of *bosonization* of field theories with fermions. Explicit soliton operators in terms of the Φ field were also first exhibited here by Mandelstam [11], an example of the later much studied *vertex* operators. The quantum theory can also be made supersymmetric [12].

(d) The discrete Z symmetry mentioned above can be seen as a *residual of the conformal invariance* of the (free) wave equation in $1 + 1$ dimensions, namely $\Phi_{tt} - \Phi_{xx} = 0$, which is Bäcklund-transformable into the conformal invariant Liouville equation $\Phi_{tt} - \Phi_{xx} = \exp \Phi$ (see e.g. [13]).

(e) The sine-Gordon theory can be also seen as the *non-linear sigma model* on the sphere S^2 [7,14], and therefore it is the simplest of the σ-models in $1 + 1$ which are exactly integrable, reflecting perhaps the fact that the naïve $0 + 1$ "σ-model", the free motion on the sphere S^n, is superintegrable [15].

All this enhances the importance of the sine-Gordon system as a toy model for some desirable properties of realistic theories such as duality, bosonization, supersymmetry and softly broken conformal invariance [13].

In this paper we focus our attention in the original motivation for the appearance of the sine-Gordon equation, namely Eq. (1.1) in light cone (characteristic) coordinates is precisely the equation which describes the classical surfaces of constant negative curvature (Enneper, ca. 1880). Since the compact representants of these surfaces are topologically classified by the genus $g > 1$, and also the manifold of solutions of Eq. (1.1) falls into classes labeled by the topological charge

$$q(\Phi) = n(\Phi) = (1/2\pi) \int \Phi_x \, \mathrm{d}x = (1/2\pi)\{\Phi(+\infty, t) - \Phi(-\infty, t)\}, \tag{1.4}$$

it is just natural to relate the two topological invariants. The resulting relation is explained in Section 3, including the concordance of the *moduli* space of these surfaces under analytic transformations with the initial positions and velocities of the n-soliton configuration. But first in Section 2, we elaborate a bit on the geometry of the equation and its solutions.

2. Surfaces of negative curvature

A surface Σ embedded in ordinary space $\mathbb{R}^3$ with *gaussian* curvature $K < 0$ everywhere has two asymptotic directions in each point, separating the regions of positive and negative *normal* curvature κ (see e.g. [3]). Taking coordinates u, v, parametrized by the arc-length, along these directions, the metric becomes

$$ds^2 = du^2 + 2F(u, v)\, du\, dv + dv^2, \tag{2.1}$$

where F is the cosine of the angle Φ of parametric lines,

$$F(u, v) = \cos \Phi(u, v), \tag{2.2}$$

all information on the surface is encoded in the function F. It is easy to prove that these coordinates can be taken throughout the surface; this is called a "Chebichef net" on the surface [16].

The gaussian curvature is easily calculated:

$$K = (1/(1 - F^2))[F_{uv} + FF_uF_v/(1 - F^2)] \tag{2.3}$$

or in terms of the Φ angle

$$\Phi_{uv} = -K \sin \Phi, \tag{2.4}$$

where K is the gaussian curvature. If K is a negative constant (e.g. $K = -1/a^2$, say), this is of course the sine-Gordon equation (1.1) in light cone (or characteristic) coordinates, for $a = 1$,

$$u = \tfrac{1}{2}(t + x), \qquad v = \tfrac{1}{2}(t - x). \tag{2.5}$$

For this reason, Eq. (2.4) was considered by Bianchi "l'equazione fondamentale di tutta la teoria delle superficie pseudospheriche" (quoted by Coleman [6]).

On the other hand the *oriented* surfaces of negative curvature, which are Riemann surfaces, are perfectly well known and classified (see e.g. [17,18]): there is the universal model, in the form of a simply connected space (with the topology of the plane), which is usually presented in three forms [17]:

(a) The Minkowski model: the upper sheet H^+ of the two-sheeted hyperboloid in $\mathbb{R}^3$ with the inherited metric from the $++-$metric in the ambient space $\mathbb{R}^3$.

(b) The Poincaré disc Δ, which is a stereographic projection of the former from the vertex of the lower hyperboloid,

$$\Delta = \{z \in C \,|\, |z| < 1\}; \qquad ds^2 = 4(dx^2 + dy^2)/(1 - |z|^2)^2. \tag{2.6}$$

(c) The upper half plane U (Klein model):

$$U = \{z \in C \mid \operatorname{Im} z > 0\}; \qquad ds^2 = (dx^2 + dy^2)/y^2. \tag{2.7}$$

$H^+ = \Delta = U$ are connected simply connected Riemann surfaces of constant negative curvature ($= -1$ by the given metric). For a detailed description of these surfaces in relation to chaotic motion see [21].

Any other Riemann surface of the conformal class $K < 0$ is obtained by quotienting by a subgroup G of the modular group M, which is a discrete automorphism group

$$\Sigma = U/G, \quad G \subset M = \mathrm{PSL}(2, Z). \tag{2.8}$$

In fact, there are *three* types of these surfaces:

(1) The simple connected case, say $\Sigma = \Delta$ or U, with the topology of the plane $C = \mathbb{R}^2$; here $G = \{e\}$.

(2) Those Σ with $G = Z = \pi_1(\Sigma)$, topology of the cylinder $S^1 \times \mathbb{R}^1$, and conformally equivalent to [18]

$$\Delta^* = \Delta - \{0\} \quad \text{or} \quad \Delta_r = \{z \in C | 0 < r < |z| < 1\}. \tag{2.9}$$

(3) All the other surfaces have a non-abelian, fundamental group π_1, are compact, and topologically homeomorphic to a sphere with g handles (or holes), where $g > 1$ ($g = 1$ corresponds to the torus T^2, which is of the conformal class flat). They can be represented as union of tori $T = T^2$,

$$\Sigma_g = T^{\#} T^{\#} \cdots {}^{\#} T \quad (g \text{ times}, g > 1), \tag{2.10}$$

where # means the *connected sum*, obtained by removing a little open disc in each torus and soldering two of them by the boundary circle [19].

(4) There remain only *non-orientable* surfaces; the compact ones are also classified by the genus g, and can be obtained by the connected sum of *projective planes* $\mathbb{R}P^2 = S^2/(Z_2)$ (antipodal map),

$$\Sigma' g = \mathbb{R}P^{2\#} \mathbb{R}P^{2\#} \cdots {}^{\#} \mathbb{R}P^2 \quad (g + 1 \text{ times}, \ g > 1). \tag{2.11}$$

The case $g = 1$ is the Klein bottle, of $K = 0$ class.

The *homology* of these surfaces is easily computed, and it is [19]

$$\begin{aligned} \chi(\Sigma_g) &= b_0 - b_1 + b_2 = 1 - 2g + 1 = 2(1 - g), \\ \chi(\Sigma'_g) &= 1 - g + 0 = 1 - g; \qquad H^1(\Sigma'_g) = Z^g + Z_2. \end{aligned} \tag{2.12}$$

In all these surfaces we can choose a constant curvature metric, by Riemann uniformization theorem [18]. However, as such they cannot be embedded in $\mathbb{R}^3$ with the induced metric from $+ + +$ (Hilbert theorem [16]).

We shall need the area of the compact oriented surfaces which might be computed from the volume element

$$dA = \sqrt{(EG - F^2)}\, du\, dv = |\sin \Phi|\, du\, dv. \tag{2.13}$$

The area and the Euler number are connected through the fundamental Gauss–Bonnet formula [16]

$$\chi = (1/2\pi)\int K\,\mathrm{d}A \tag{2.14}$$

This will be the key to identify the particular Riemann surfaces.

3. Genus from soliton configurations

The solutions of sine-Gordon equation (1.1) or (2.4) are classified by the topological charge (1.4), namely:

(a) $q = 0$, the vacuum sector. It contains the vacuum solution $\Phi = 2\pi k$, the soliton–antisoliton scattering configurations, the soliton–antisoliton bound states (breather mode) and combinations thereof.

(b) $q = \pm 1$; it contains one soliton (resp. antisoliton) of Eq. (1.3), translated and/or boosted, plus any solution of (a) above.

(c) q arbitrary integer; this is the multisoliton configuration; for example $q = +2$ will contain two solitons plus any solution of (a) above; etc.

Which negative curvature surfaces do these configurations belong to? Let us start with the $q = 2$ two soliton states. The connection with the genus of the potential surface will be made through the Gauss–Bonnet theorem: we can perform an area integration, applying (2.13) and (2.14) to Eq. (2.4) $\sin\Phi = \partial_{uv}\Phi$:

$$\begin{aligned}\int K\,\mathrm{d}A &= (-1)\int |\sin\Phi|\,\mathrm{d}u\,\mathrm{d}v = (-1)\int \partial_{uv}\Phi\,\mathrm{d}u\,\mathrm{d}v\\ &= (-1)\Phi \quad \text{(boundaries)}.\end{aligned} \tag{3.1}$$

This is called "Hazzidaki's formula" [16]. Now for the two-soliton configuration it turns out that the algebraic sum of the boundary values are just the jump in $x = \pm\infty$ at $t = 0$ minus the jump in $t = \pm\infty$ at $x = 0$, due to the relation between u, v and x, t (2.5):

$$\begin{aligned}\text{Area} &= \Phi(x = +\infty, t = 0) - \Phi(x = -\infty, t = 0)\\ &\quad - \{\Phi(x = 0, t = +\infty) - \Phi(x = 0, t = -\infty)\},\end{aligned} \tag{3.2}$$

which is just 4π for the two-soliton state (there is no jump in t, as $\Phi(x = 0, t)$= const. by symmetry): The explicit formula for the two-soliton solution in x, t coordinates is [1,4]:

$$\Phi(x,t) = 4\,\mathrm{arctg}\,\{-(\cosh\gamma vt)/v\sinh(\gamma x)\}, \tag{3.3}$$

where v is the relative velocity and $\gamma^2 = (1 - v^2)^{-1}$.

Therefore we are describing a $g = 2$ surface, according to the integral formula

$$\chi = (1/2\pi)\int K\,\mathrm{d}A = 2(1-g) = (-1)4\pi/2\pi, \quad g = 2. \tag{3.4}$$

The generalization to $n = 2\nu$ *even* number of solitons is immediate, because again there is no jump in t while the jump in x is given by $2\pi n$:

$$2(1-g) = -2\pi(2\nu)/2\pi \quad \Rightarrow \quad g = \nu + 1 \quad (\nu > 0) \tag{3.5}$$

and it describes the compact oriented surface of genus g. The span in Φ, namely $4\pi\nu$, reflects the "holes" of the surface. This is a satisfactory result. Of course, the integration can be performed analytically also. The configuration with $2n$ antisolitons will presumably describe the same surface with changed orientation.

The correspondence goes along also with the integration parameters; namely we can choose *three* integration constants for each soliton, the center, origin of time and velocity (i.e. the three parameters of the 1+1 Poincaré group). But the *moduli space* of the surface of genus g is known to be the Teichmüller space [17,18] of *complex* dimension (an intuitive derivation of the moduli space formula for Riemann surfaces is given in the book by Witten et al. [20])

$$B_g = \dim\ \text{Teich}\,(\Sigma_g) = 3(g-1) = (1/2)3 \times 2\nu = (1/2) \quad [\#\ \text{real param.}] \tag{3.6}$$

because $g = \nu + 1$. So this is again in agreement.

We do not have a satisfactory answer for the odd-soliton case, for which the integration is ill-defined. If we maintain (3.4) for any soliton number, i.e.

$$-\chi = n \Longrightarrow g = n + 1 \quad \text{for } \textit{unoriented} \text{ surfaces}, \quad \chi = 1 - g \tag{3.7}$$

and we conjecture that this is true; in this case the soliton will describe the "unoriented" pretzel, $g = 2$. This goes on with the fact that the soliton will be a fermion, and fermions are odd under full rotations. Again, the concordance goes also with the moduli space, for which the freedom is now in *real* dimension [17]

$$B_g = 3(g-1) = \#\ \text{param. of the } n - \text{soln. confign.} \tag{3.8}$$

There are still other surfaces (Section 2); we conjecture also that the breather mode, i.e. a non-trivial solution with $q = 0$, will correspond to the non-compact case, e.g. to the simply connected model Δ or U.

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